3. Integracija pomoću smjene promjenjivih

Veoma efikasna metoda integriranja je metoda pomoću smjene promjenjivih, a rezultat metode je da se dati integral zamjeni drugim integralom.

Posmatrajmo dati integral $\int f(x)\,dx$. Ako je moguće, želimo promjenjivu x zamjeniti nekom novom promjenjivom t, koristeći smjenu $x=\varphi(t)$. Tada je $dx=\varphi'(t)\,dt$ pa imamo

$$\int f(x)\,dx = \int f[\varphi(t)]\,\varphi'(t)\,dt = \int F(t)\,dt.$$

Na ovaj način se često zadani integral svodi na elementarni tablični integral. Na primjer, želimo odrediti integral $J=\int \frac{dx}{1+\sqrt{x}}$, uvođenjem smjene $x=t^2$. Tada je $dx=2t\,dt$ pa imamo

$$J=\int \frac{2t\,dt}{1+t} = 2\int \frac{t+1-1}{t+1}\,dt = 2\int\left(1-\frac{1}{t+1}\right)dt =$$

$$=2\int dt - 2\int \frac{dt}{t+1} = 2t-2\ln(t+1)+C=$$

$$=2\sqrt{x}-2\ln(\sqrt{x}+1)+C$$

Isti integral smo mogli odrediti i uvođenjem smjene $t=1+\sqrt{x}$ iz čega slijedi $x=(t-1)^2$, $dx=2(t-1)\,dt$

$$J=\int \frac{2(t-1)\,dt}{t} = 2\int\left(1-\frac{1}{t}\right)dt = 2t-2\ln t+C = 2(1+\sqrt{x})-2\ln(1+\sqrt{x})+C$$

Odrediti integrale

a) $\int \frac{2x\,dx}{x^4+3}$ b) $\int \frac{\sin x\,dx}{\sqrt{1+2\cos x}}$ c) $\int \frac{x\,dx}{\sqrt[3]{x^2+a}}$

d) $\int \frac{\sqrt{1+\ln x}}{x}\,dx$ e) $\int \frac{dy}{\sqrt{e^y+1}}$ f) $\int \frac{dt}{\sqrt{(1-t^2)^3}}$

Rj.

a) $$\int \frac{2x\,dx}{x^4+3} = \left| \begin{array}{l} x^2=t \\ 2x\,dx=dt \end{array} \right| = \int \frac{dt}{t^2+3} = \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{t}{\sqrt{3}} + C =$$

$$= \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{x^2}{\sqrt{3}} + C$$

b) $$\int \frac{\sin x\,dx}{\sqrt{1+2\cos x}} = \left| \begin{array}{l} 1+2\cos x = t \\ -2\sin x\,dx = dt \\ \sin x\,dx = -\frac{1}{2}\,dt \end{array} \right| = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\frac{1}{2}\int t^{-1/2}\,dt$$

$$= -\frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = C - \sqrt{t} = C - \sqrt{1+2\cos x}$$

c) $$\int \frac{x\,dx}{\sqrt[3]{x^2+a}} = \left| \begin{array}{l} x^2+a=z \\ 2x\,dx=dz \\ x\,dx=\frac{1}{2}\,dz \end{array} \right| = \frac{1}{2}\int \frac{dz}{\sqrt[3]{z}} = \frac{1}{2}\int z^{-\frac{1}{3}}\,dz =$$

$$= \frac{1}{2}\cdot\frac{z^{\frac{2}{3}}}{\frac{2}{3}} + C = \frac{1}{2}\cdot\frac{3}{2}\cdot\sqrt[3]{z^2} + C = \frac{3}{4}\sqrt[3]{(x^2+a)^2} + C$$

d) $$\int \frac{\sqrt{1+\ln x}}{x}\,dx = \left| \begin{array}{l} 1+\ln x = v \\ \frac{1}{x}\,dx = dv \end{array} \right| = \int \sqrt{v}\,dv = \int v^{\frac{1}{2}}\,dv = \frac{v^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{2}{3}\sqrt{v^3} + C = \frac{2}{3}\sqrt{(1+\ln x)^3} + C$$

e) $\displaystyle\int \frac{dy}{\sqrt{e^y+1}} = \left| \begin{array}{ll} e^y+1=t^2 & (t^2-1)\,dy = 2t\,dt \\ e^y = t^2-1 & dy = \dfrac{2t\,dt}{t^2-1} \\ de^y = d(t^2-1) & \\ e^y dy = 2t\,dt & \end{array} \right| =$

$$= \int \frac{\frac{2t\,dt}{t^2-1}}{\sqrt{t^2}} = \int \frac{2t\,dt}{t(t^2-1)} = 2\int \frac{dt}{t^2-1} = 2\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C$$

$$= \ln \frac{\sqrt{e^y+1}-1}{\sqrt{e^y+1}+1} + C$$

f) $\displaystyle\int \frac{dt}{\sqrt{(1-t^2)^3}} = \left| \begin{array}{l} t=\sin\varphi \\ dt=\cos\varphi\,d\varphi \\ 1-t^2 = 1-\sin^2\varphi = \cos^2\varphi \end{array} \right| =$

$$= \int \frac{\cos\varphi\,d\varphi}{\sqrt{\cos^6\varphi}} = \int \frac{\cos\varphi\,d\varphi}{\cos^3\varphi} = \int \frac{d\varphi}{\cos^2\varphi} = \operatorname{tg}\varphi + C$$

$$= \frac{\sin\varphi}{\cos\varphi} + C = \frac{\sin\varphi}{\sqrt{1-\sin^2\varphi}} + C = \frac{t}{\sqrt{1-t^2}} + C$$

Zadaci za vježbu

Izračunati sljedeće integrale i provjeriti rezultat diferenciranjem:

1. $\int \frac{x^2\,dx}{5-x^6}$. Pomoću smjene $t=x^3$.

2. $\int \frac{e^x\,dx}{3+4e^x}$. Pomoću smjene $z=3+4e^x$.

3. $\int tg^3\varphi\,d\varphi$. Pomoću smjene $\varphi = arc\,tg\,t$.

4. $\int x^3\sqrt{a-x^2}\,dx$. Pomoću smjene $\sqrt{a-x^2}=z$.

5. $\int \frac{x^2-x}{(x-2)^3}\,dx$. Pomoću smjene $x-2=t$

6. $\int x\sqrt{a-x}\,dx$. Pomoću smjene $a-x=t^2$

7. * $\int \frac{dx}{x\sqrt{1+x^2}}$. Pomoću smjene $x=\frac{1}{t}$.

8. * $\int \frac{dx}{\sin 2x}$. Pomoću smjene $tg\,x = z$.

Odrediti integrale

9. $\int \frac{x\,dx}{\sqrt{x^4+1}}$

10. $\int \frac{\sqrt{x}\,dx}{1+\sqrt{x}}$

11. $\int \frac{e^{2x}\,dx}{e^x-1}$

12. $\int \frac{dx}{x\ln x}$

13. $\displaystyle\int \frac{\cos x \, dx}{\sqrt{1+2\sin^2 x}}$

14. $\displaystyle\int \frac{\sin 2x \, dx}{\sqrt{2+\cos^2 x}}$

15.* $\displaystyle\int \frac{e^{2x} \, dx}{\sqrt[4]{1+e^x}}$

16.* $\displaystyle\int \frac{\sqrt{x} \, dx}{1+\sqrt[4]{x^3}}$

Rješenja:

1. $\displaystyle\frac{1}{6\sqrt{5}} \ln \left| \frac{x^3+\sqrt{5}}{x^3-\sqrt{5}} \right|$

2. $\displaystyle\frac{1}{4} \ln (3+4e^x)$

3. $\displaystyle\frac{1}{2} \operatorname{tg}^2 \varphi + \ln |\cos \varphi|$

4. $\displaystyle-\frac{3x^2+2a}{15} \sqrt{(a-x^2)^2}$

5. $\displaystyle\ln |x-2| - \frac{3x-5}{(x-2)^2}$

6. $\displaystyle\frac{2}{15} (3x^2-ax-2a^2) \sqrt{a-x}$

7. $\displaystyle\pm \ln \frac{x}{1 \pm \sqrt{1+x^2}}$, gdje je "+" ako $x>0$, "−" ako je $x<0$

ili drugačije $\displaystyle\ln \frac{|x|}{1+}$

8. $\displaystyle\frac{1}{2} \ln |\operatorname{tg} x|$

9. $\displaystyle\frac{1}{2} \ln (x^2+\sqrt{1+x^4})$

10. $x-2\sqrt{x}+2\ln(1+\sqrt{x})$

11. $e^x + \ln |e^x - 1|$

12. $\ln |\ln x|$

13. $\displaystyle\frac{1}{\sqrt{2}} \ln \left(\sin x + \sqrt{\frac{1}{2} + \sin^2 x} \right)$

14. $-2\sqrt{2+\cos^2 x}$

15. $\displaystyle\frac{4}{21} (3e^x - 4) \sqrt[4]{(e^x+1)^3}$

16. $\displaystyle\frac{4}{3} \left(\sqrt[4]{x^3} - \ln \left(1 + \sqrt[4]{x^3}\right) \right).$

Izabrani **Zadaci za vježbu** sa rješenjima
(iz lekcije Integracija pomoću zamjene promjenjivih)

1. $$\int \frac{dx}{2x+5} = \left| \begin{array}{l} t=2x+5 \\ dt=2\,dx \\ dx=\frac{dt}{2} \end{array} \right| = \int \frac{\frac{dt}{2}}{t} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t|+C = \frac{1}{2}\ln|2x+5|+C$$

2. $$\int \sin(4x+1)\,dx = \left| \begin{array}{l} 4x+1=t \\ 4dx=dt \\ dx=\frac{dt}{4} \end{array} \right| = \int \sin t \cdot \frac{dt}{4} = \frac{1}{4}\int \sin t\,dt = -\frac{1}{4}\cos t + C =$$
$$= -\frac{1}{4}\cos(4x+1)+C$$

3. $$\int (3x-1)^9\,dx = \left| \begin{array}{l} 3x-1=t \\ 3dx=dt \\ dx=\frac{dt}{3} \end{array} \right| = \int t^9 \cdot \frac{dt}{3} = \frac{1}{3}\int t^9\,dt = \frac{1}{3}\cdot\frac{t^{10}}{10}+C =$$
$$= \frac{(3x-1)^{10}}{30}+C$$

4. $$\int e^{1-3x}\,dx = \left| \begin{array}{l} 1-3x=t \\ -3dx=dt \\ dx=-\frac{dt}{3} \end{array} \right| = \int e^t \cdot -\frac{dt}{3} = -\frac{1}{3}\int e^t\,dt = -\frac{1}{3}e^t+C =$$
$$-\frac{1}{3}e^{1-3x}+C$$

5. $$\int \frac{dx}{\sqrt{1-(3x+2)^2}} = \left| \begin{array}{l} 3x+2=t \\ 3dx=dt \\ dx=\frac{dt}{3} \end{array} \right| = \frac{1}{3}\int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{3}\arcsin t + C =$$
$$= \frac{1}{3}\arcsin(3x+2)+C$$

6. $$\int \cos(6x+4)\,dx \qquad \text{Rj.} \quad \frac{1}{6}\sin(6x+4)+C$$

7. $$\int \frac{dx}{\cos^2(7x+8)} \qquad \text{Rj.} \quad \frac{1}{7}\cdot\frac{\sin(7x+8)}{\cos(7x+8)}$$

8. $$\int \frac{dx}{1+(5x-2)^2} \qquad \text{Rj.} \quad \frac{1}{5}\operatorname{arctg}(5x-2)$$

9.

$$\int \frac{dx}{4x^2+9} = \int \frac{dx}{(2x)^2+3^2} = \left| \begin{array}{l} 2x = 3t \\ 2dx = 3dt \\ dx = \frac{3}{2}dt \\ t = \frac{2x}{3} \end{array} \right| = \frac{3}{2}\int \frac{dt}{(3t)^2+3^2} = \frac{3}{2}\int \frac{dt}{9t^2+9} =$$

$$= \frac{3}{2}\cdot\frac{1}{9}\int \frac{dt}{t^2+1} = \frac{1}{6}\,\mathrm{arctg}\, t + c = \frac{1}{6}\,\mathrm{arctg}\,\frac{2x}{3} + c$$

10.

$$\int \frac{dx}{\sqrt{2x^2+25}} = \int \frac{dx}{\sqrt{(\sqrt{2}x)^2+5^2}} = \left| \begin{array}{l} \sqrt{2}x = 5t \\ \sqrt{2}\,dx = 5dt \\ dx = \frac{5}{\sqrt{2}}dt \\ t = \frac{\sqrt{2}}{5}x \end{array} \right| = \frac{5}{\sqrt{2}}\int \frac{dt}{\sqrt{\underbrace{25t^2+25}_{25\cdot(t^2+1)}}} =$$

$$= \frac{5}{\sqrt{2}}\cdot\frac{1}{5}\int \frac{dt}{\sqrt{t^2+1}} = \frac{1}{\sqrt{2}}\cdot \ln\left|t+\sqrt{t^2+1}\right| + c = \frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2}}{5}x + \sqrt{\frac{2}{25}x^2+1}\right| + c$$

11.

$$\int \frac{dx}{5x^2-49} = \int \frac{dx}{(\sqrt{5}x)^2-7^2} = \left| \begin{array}{l} \sqrt{5}x = 7t \\ \sqrt{5}dx = 7dt \\ dx = \frac{7}{\sqrt{5}}dt \\ t = \frac{\sqrt{5}x}{7} \end{array} \right| = \frac{7}{\sqrt{5}}\int \frac{dt}{49t^2-49} = \frac{7}{\sqrt{5}}\cdot\frac{1}{49}\int\frac{dt}{t^2-1}$$

$$= \frac{1}{7\sqrt{5}}\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + c = \frac{1}{14\sqrt{5}}\ln\left|\frac{\frac{\sqrt{5}}{7}x-1}{\frac{\sqrt{5}}{7}x+1}\right| + c = \frac{1}{14\sqrt{5}}\ln\left|\frac{\sqrt{5}x-7}{\sqrt{5}x+7}\right|$$

12.

$$\int \frac{dx}{\sqrt{7-9x^2}} = \int \frac{dx}{\sqrt{(\sqrt{7})^2-(3x)^2}} = \left| \begin{array}{l} 3x = \sqrt{7}t \\ 3dx = \sqrt{7}dt \\ dx = \frac{\sqrt{7}}{3}dt \\ t = \frac{3x}{\sqrt{7}} \end{array} \right| = \frac{\sqrt{7}}{3}\int \frac{dt}{\sqrt{7-7t^2}} = \frac{\sqrt{7}}{3}\cdot\frac{1}{\sqrt{7}}\int\frac{dt}{\sqrt{1-t^2}}$$

$$= \frac{1}{3}\arcsin t + c = \frac{1}{3}\arcsin\left(\frac{3x}{\sqrt{7}}\right) + c$$

13. $\displaystyle\int \frac{dx}{4x^2+11}$, Rj. $\displaystyle\frac{\sqrt{11}}{22}\,\mathrm{arctg}\,\frac{2\sqrt{11}x}{11} + c$

14. $\displaystyle\int \frac{dx}{\sqrt{9x^2-16}}$, Rj. $\displaystyle\frac{1}{3}\ln\left|3x+\sqrt{9x^2-16}\right| + c$

15. $\displaystyle\int \frac{dx}{\sqrt{2x^2+5}}$, Rj. $\displaystyle\frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{10}}{5}x+\sqrt{\frac{2}{5}x^2+1}\right| + c$

16. $$\int \operatorname{ctg} x\, dx = \int \frac{\cos x}{\sin x}\, dx = \left| \begin{matrix} \sin x = t \\ \cos x\, dx = dt \end{matrix} \right| = \int \frac{dt}{t} = \ln|t| + C =$$
$$= \ln|\sin x| + C$$

17. $$\int \frac{3x^2+4x-4}{x^3+2x^2-4x+6}\, dx = \left| \begin{matrix} x^3+2x^2-4x+6 = t \\ (3x^2+4x-4)\, dx = dt \end{matrix} \right| = \int \frac{dt}{t} = \ln|t| + C =$$
$$= \ln|x^3+2x^2-4x+6| + C$$

18. $$\int \frac{x-5}{\sqrt{x^2-10x+7}}\, dx = \left| \begin{matrix} x^2-10x+7 = t \\ (2x-10)\, dx = dt \\ (x-5)\, dx = \frac{dt}{2} \end{matrix} \right| = \frac{1}{2} \int \frac{dt}{\sqrt{t}} = \frac{1}{2} \cdot \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C =$$
$$= \frac{1}{2} \cdot 2\sqrt{t} + C = \sqrt{x^2-10x+7} + C$$

19. $$\int \frac{x-3}{x^2-6x+7}\, dx = \left| \begin{matrix} x^2-6x+7 = t \\ (2x-6)\, dx = dt \\ (x-3)\, dx = \frac{dt}{2} \end{matrix} \right| = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \ln|t| + C$$
$$= \frac{1}{2} \ln|x^2-6x+7| + C$$

20. $$\int \frac{x^3\, dx}{\sqrt{x^4+1}} = \left| \begin{matrix} x^4+1 = t \\ 4x^3\, dx = dt \\ x^3\, dx = \frac{dt}{4} \end{matrix} \right| = \frac{1}{4} \int \frac{dt}{\sqrt{t}} = \frac{1}{4} \cdot \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{1}{4} \cdot 2\sqrt{t} + C$$
$$= \frac{1}{2}\sqrt{x^4+1} + C$$

21. $\int \frac{3x^2}{\sqrt{x^3-2}}\, dx$ Rj. $2\sqrt{x^3-2} + C$

22. $\int \operatorname{tg} x\, dx$ Rj. $-\ln|\cos x| + C$

23. $\int \frac{\sin x}{\sqrt{5\cos x - 2}}\, dx$ Rj. $-\frac{2}{5}\sqrt{5\cos x - 2} + C$

24. $$\int e^{\cos x}\cdot \sin x\, dx = \left| \begin{array}{l} \cos x = t \\ -\sin x\, dx = dt \\ \sin x\, dx = -dt \end{array} \right| = \int e^{t}\cdot(-dt) = -\int e^{t} dt =$$

$$= -e^{t} + c = -e^{\cos x} + c$$

25. $$\int \frac{dx}{x\sqrt[5]{\ln x}} = \left| \begin{array}{l} \ln x = t^5 \\ \frac{1}{x}\, dx = 5t^4 dt \\ t = \sqrt[5]{\ln x} \end{array} \right| = \int \frac{5t^4 dt}{t} = 5\int t^3 dt = 5\cdot \frac{t^4}{4} + c$$

$$= \frac{5}{4}\sqrt[5]{\ln^4 x} + c$$

26. $$\int \frac{x^3 dx}{x^8 - 2} = \int \frac{x^3\, dx}{(x^4)^2 - 2} = \left| \begin{array}{l} x^4 = t \\ 4x^3 dx = dt \\ x^3 dx = \frac{1}{4}\, dt \end{array} \right| = \frac{1}{4}\int \frac{dt}{t^2 - 2} = \frac{1}{4}\cdot\frac{1}{2\sqrt{2}}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| + c$$

$$= \frac{1}{8\sqrt{2}}\ln\left|\frac{x^4 - \sqrt{2}}{x^4 + \sqrt{2}}\right| + c$$

27. $$\int \frac{\sqrt[3]{\operatorname{tg} x}}{\cos^2 x}\, dx = \left| \begin{array}{l} \operatorname{tg} x = t^3 \\ \frac{1}{\cos^2 x}\, dx = 3t^2 dt \end{array} \right| = \int \sqrt[3]{t^3}\cdot 3t^2 dt = 3\int t^3 dt =$$

$$= 3\,\frac{t^4}{4} + c = \frac{3}{4}\sqrt[3]{\operatorname{tg}^4 x} + c$$

28. $$\int x(1-x)^{10} dx = \left| \begin{array}{l} 1-x = t \\ -dx = dt \\ dx = -dt \\ x = 1-t \end{array} \right| = \int (1-t)\, t^{10}\cdot(-dt) = -\int (t^{10} - t^{11})\, dt$$

$$= -\frac{t^{11}}{11} + \frac{t^{12}}{12} + c = \frac{(1-x)^{12}}{12} - \frac{(1-x)^{11}}{11} + c$$

29. $$\int \frac{x^2 - 1 \;\; /:x^2}{x^4 + 1 \;\; /:x^2}\, dx = \int \frac{1 - \frac{1}{x^2}}{x^2 + \frac{1}{x^2}}\, dx =$$

treba da izberemo zamjenu takvu da dobijemo $\left(1 - \frac{1}{x^2}\right) dx = dt$

$$t = x + \frac{1}{x} \Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dt$$

$$\left(x + \frac{1}{x}\right)^2 = t^2 \Rightarrow x^2 + 2x\cdot\frac{1}{x} + \frac{1}{x^2} = t^2$$

$$x^2 + \frac{1}{x^2} = t^2 - 2$$

$$= \int \frac{dt}{t^2 - 2} = \frac{1}{2\sqrt{2}}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| + c = \frac{1}{2\sqrt{2}}\ln\left|\frac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}}\right| + c$$

30. $$\int \sqrt{\frac{\arcsin x}{1-x^2}}\,dx = \int \frac{\sqrt{\arcsin x}}{\sqrt{1-x^2}}\,dx = \left| \begin{array}{l} \arcsin x = t^2 \\ \frac{dx}{\sqrt{1-x^2}} = 2t\,dt \\ t = \sqrt{\arcsin x} \end{array} \right| =$$

$$= \int \sqrt{t^2}\, 2t\,dt = 2\int t^2\,dt = 2\cdot\frac{t^3}{3} + C = \frac{2}{3}\sqrt{\arcsin^3 x} + C$$

31. $$\int (x+4)\sqrt[5]{2x-1}\,dx = \left| \begin{array}{ll} 2x-1 = t^5 & 2x = t^5+1 \\ 2\,dx = 5t^4\,dt & x = \frac{t^5+1}{2} \\ dx = \frac{5}{2}t^4\,dt & t = \sqrt[5]{2x-1} \end{array} \right| = \int \left(\frac{t^5+1}{2}+4\right)\sqrt[5]{t^5}\cdot\frac{5}{2}t^4\,dt$$

$$= \frac{5}{2}\int \frac{t^5+1+8}{2}\,t^5\,dt = \frac{5}{4}\int (t^5+9)t^5\,dt = \frac{5}{4}\int (t^{10}+9t^5)\,dt =$$

$$= \frac{5}{4}\cdot\frac{t^{11}}{11} + \frac{5}{4}\cdot \overset{3}{\cancel{9}}\cdot\frac{t^6}{\underset{2}{\cancel{6}}} + C = \frac{5}{44}\sqrt[5]{(2x-1)^{11}} + \frac{15}{8}\sqrt[5]{(2x-1)^6} + C$$

32. ISPITNI

$$\int \frac{\ln x\,dx}{x\sqrt{1+\ln x}} = \left| \begin{array}{l} \ln x = t \\ \frac{1}{x}dx = dt \end{array} \right| = \int \frac{t+1-1}{\sqrt{1+t}}\,dt = \int \frac{t+1}{\sqrt{1+t}}\cdot\frac{\sqrt{1+t}}{\sqrt{1+t}}\,dt - \int \frac{dt}{\sqrt{1+t}}$$

$$= \int (1+t)^{\frac{1}{2}}\,dt - \int (1+t)^{-\frac{1}{2}}\,dt = \frac{(1+t)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(1+t)^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{2}{3}(1+t)^{\frac{3}{2}} - 2(1+t)^{\frac{1}{2}} + C$$

$$= \frac{2}{3}(1+t)^{\frac{1}{2}}\cdot\left((1+t)-3\right) + C = \frac{2}{3}\sqrt{1+\ln x}\,(\ln x - 2) + C$$

33. ✓ ISPITNI

$$\int \frac{e^{3x}(10-2e^{3x})}{2e^{6x}-10e^{3x}+12}\,dx$$

Rj. $-\frac{1}{6}\ln\left|e^{6x}-5e^{3x}+6\right| + \frac{5}{6}\ln\left|\frac{e^{3x}-3}{e^{3x}-2}\right| + C$

34. ✓ $$\int \frac{1}{x^2}\sin\frac{1}{x}\,dx$$

36. ✓ $$\int \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$$

35. ✓ $$\int x(2x+3)^7\,dx$$

37. ✓ $$\int \frac{x^2+1}{\sqrt{x^6-7x^4+x^2}}\,dx$$

(#) Izračunati integral $\int \sqrt{\frac{x-2}{x+2}}\,dx$.

Rj.
$$\int \sqrt{\frac{x-2}{x+2}}\,dx = \int \frac{\sqrt{x-2}}{\sqrt{x+2}}\,dx = \int \frac{\sqrt{x-2}\cdot\sqrt{x-2}}{\sqrt{x+2}\cdot\sqrt{x-2}}\,dx = \int \frac{x-2}{\sqrt{x^2-4}}\,dx$$
$$= \int \frac{x}{\sqrt{x^2-4}}\,dx - 2\int \frac{dx}{\sqrt{x^2-4}}$$

$$\int \frac{x}{\sqrt{x^2-4}}\,dx = \left| \begin{array}{l} x^2-4=t \\ 2x\,dx = dt \\ x\,dx = \frac{1}{2}dt \end{array} \right| = \frac{1}{2}\int \frac{dt}{\sqrt{t}} = \frac{1}{2}\int t^{-\frac{1}{2}}dt = \frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= \sqrt{t} + c = \sqrt{x^2-4} + c$$

$$\int \frac{dx}{\sqrt{x^2-4}} = \left| \begin{array}{l} x = 2s \\ dx = 2\,ds \\ s = \frac{1}{2}x \end{array} \right| = \int \frac{2\,ds}{\sqrt{4s^2-4}} = \frac{2}{\sqrt{4}}\int \frac{ds}{\sqrt{s^2-1}} = \ln\left|s + \sqrt{s^2-1}\right| + C_1$$
$$= \ln\left|\frac{1}{2}x + \sqrt{\frac{1}{4}x^2 - 1}\right| + C_1 = \ln\left|\frac{1}{2}x + \frac{1}{2}\sqrt{x^2-4}\right| + C_1$$
$$= \ln\frac{1}{2} + \ln\left|x + \sqrt{x^2-4}\right| + C_1 = \ln\left|x + \sqrt{x^2-4}\right| + C$$

$$\int \sqrt{\frac{x-2}{x+2}}\,dx = \sqrt{x^2-4} - 2\ln\left|x + \sqrt{x^2-4}\right| + C$$

\# Izračunati integral $\int x^3\sqrt{1+a^2x^2}\,dx$, $(a>0)$.

Rj.

$$\int x^3\sqrt{1+a^2x^2}\,dx=\int x^2\cdot x\cdot\sqrt{1+a^2x^2}\,dx=\left|\begin{array}{c}1+a^2x^2=t^2\\ a^2\cdot 2x\,dx=2t\,dt\\ x\,dx=\frac{1}{a^2}t\,dt\\ a^2x^2=t^2-1\\ x^2=\frac{1}{a^2}(t^2-1)\end{array}\right|$$

$$=\int\frac{1}{a^2}(t^2-1)\cdot\frac{1}{a^2}t\cdot t\,dt=$$

$$=\frac{1}{a^4}\int(t^4-t^2)\,dt=\frac{1}{a^4}\cdot\frac{1}{5}t^5-\frac{1}{a^4}\cdot\frac{1}{3}t^3=$$

$$=\frac{1}{5a^4}\sqrt{(1+a^2x^2)^5}-\frac{1}{3a^4}\sqrt{(1+a^2x^2)^3}+C$$

\# Izračunati integral $\int\frac{dx}{\sqrt[4]{x^3}\,(1+\sqrt[6]{x})}$.

Rj.

$$\int\frac{dx}{\sqrt[4]{x^3}\,(1+\sqrt[6]{x})}=\left|\begin{array}{c}x=t^{12}\quad t=\sqrt[12]{x}\\ dx=12t^{11}dt\\ \sqrt[4]{x^3}=\sqrt[4]{t^{36}}=t^9\\ \sqrt[6]{x}=\sqrt[6]{t^{12}}=t^2\end{array}\right|=\int\frac{12t^{11}dt}{t^9(1+t^2)}=$$

$$=12\int\frac{t^2\ {}^{+1-1}}{1+t^2}dt=12\int\frac{t^2+1}{t^2+1}dt-12\int\frac{dt}{1+t^2}=$$

$$=12t-12\,\mathrm{arctg}\,t+C=12\sqrt[12]{x}-12\,\mathrm{arctg}\sqrt[12]{x}+C$$